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NOTE ON EXIT TIME FOR NONLINEAR PAST-DEPENDENT RECURSIONS

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Abstract

In present work studied the AR(1) stochastic model with recurrent relations and small "noise". It is proved the estimate for exit times.

Keywords

Large deviation principle, Autoregressive process, Gaussian distribution

1.Introduction

The present paper deals with a small stochastic perturbation of a nonlinear one-dimensional dynamic system of the form

$x_{n+1} = f_{n+1}(x_n), x_0 \in [-1,1], n \in N$ (1.1)

where the map $f : \mathbb{R} \to [-1,1]$ has a unique stable fixed point $x_0 = 0$ .An interesting and well studied example is logistic family $f_\alpha(x) = \alpha x(\text{mod}1), \alpha \in [0,1], x \in (0,1)$ (see for instance {Collet Ekhmann On iterated…}). This family was studied by dynamical renormalisation group method . The renormalisation group method in dynamical systems first used by M. Feigenbaum in his universality theory {see [7], [8]}. Notice that the many problems of population theory of biology can be reduced to study the dynamical systems of type (1.1) (see for instance [9]). To study the small stochastic perterbutions is one of fundamental problems of the theory of stochastic persecutions of dynamical systems (see [2], [4]).

An interesting problem is to investigate the exit time for a stochastic process $Y_n, n > 0$.

Klebaner and Liptser in [1] proved a large deviationprinciple (LDP) for a class of past-dependent models. As an example, they used the univariate

autoregressive process $X_n^{(\varepsilon)}, n \geq 1$ defined as

$X_{n+1}^{(\varepsilon)} = f\left(X_n^{(\varepsilon)}\right) + \varepsilon\xi_{n+1}$ (1.2)

where a contractive $f(x)$ is continuous function on $\mathbb{R}^1$ and ε is positive parameter, $\{\xi_n, n \geq 1\}$ is an i.i.d. sequences of standard normal random variables.

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The process $X_n^{(\varepsilon)}$ has a stationary distribution which is normal with mean 0 and variance $\frac{1}{1-a^2}$. Klebaner and Liptser showed that the family of processes $X_n^{(\varepsilon)}$ obeys an LDP with rate of speed ε^2 and rate function

$$I(\underline{u}) = \begin{cases} \frac{1}{2}\sum_{t=1}^{\infty}(u_t - au_{t-1})^2, & u_0 = x_0, \\ \infty, & otherwise, \end{cases}$$

For each $\varepsilon \in (0, \varepsilon_0]$ we define the exit function by

$$\tau^{(\varepsilon)} := \min\left\{n \ge 1 : \left|X_n^{(\varepsilon)}\right| \ge 1\right\}$$

Klebaner and Liptser in [1] applying large deviationprinciple (LDP) proved that

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log E_{x_0}\tau \le \frac{1}{2}\left(1-a^2\right).$$

G.Hognas and B.Jung in [6] investigated the estimates for the exit times in the case some piecewise contractive function f .Also , it is proved for AR(1) with contractive function $f_a(x) := ax^2$ with parameter $a \in (0,1)$ the following bounds

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log E_{x_0}\tau \le \frac{1}{2}\left(1-a^2\right), \text{ if } 0 < a < \frac{1}{2},$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log E_{x_0}\tau \le \frac{1}{2}\left(\frac{1}{a} - \frac{1}{4a^2}\right), \text{ if } \frac{1}{2} \le a.$$

In present work we study the stochastic sequence defined by past-dependent recursion with small noise

$$\begin{cases} X_{n+1}^{(\varepsilon)} = a\left|\sin\left(\frac{\pi}{2}X_n^{(\varepsilon)}\right)\right| + \varepsilon\xi_{n+1}, \\ X_0 = 0. \end{cases}$$

where the parameters $\varepsilon > 0, a \in [0,1]$ and $\{\xi_n, n \ge 1\}$ is an i.i.d. sequence of standard Gaussian random variables with parameters (0,1).

Next we formulate the main result of our paper.

Theorem 1.1. Let $\varepsilon \in (0, \varepsilon_0]$, $a \in (0,1)$. Consider the stochastic sequence $\left\{X_n^{(\varepsilon)}, n \ge 1\right\}$ defined by (1.3) and the exit time $\tau_a = \min\left\{k : \left|X_k^{(\varepsilon)}\right| \ge 1\right\}$. Then there exists a number $0 < P_a < \frac{1}{2}$,such that

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$$\overline{\lim_{\varepsilon\to 0}}\varepsilon^2 \log E\tau_a \le P_a$$

2. Large deviation principle for a past-dependent stochastic process

Following Varadhan [3], family $\left(X^{\varepsilon}\right)_{k\ge m}$ is said to satisfy the LDP in the metric space $\left(R^{\infty},\rho\right)$ with the rate of speed $q(\varepsilon)$ and the **rate function** $J(u)$ if

(0) there exists a function $q(\varepsilon)$ $J = J(\underline{u}), \underline{u} = (u_1, u_2, \ldots) \in R^{\infty}$ which takes values in $[0,\infty]$ such that for every $\alpha \ge 0$ the set $\Phi(\alpha) = \left\{\underline{u} \in R^{\infty} : J(\underline{u}) \le \alpha\right\}$ is compact in $\left(\mathbb{R}^{\infty}, \rho_{\infty}\right)$;

(1) For every closed set $F \in \left(\mathbb{R}^{\infty}, \rho_{\infty}\right)$

$$\overline{\lim_{\varepsilon\to 0}} q(\varepsilon)\log P\left(\left(X_n^{(\varepsilon)}\right)_{n\ge m} \in F\right) \le -\inf_{\underline{u}\in F} J(\underline{u});;$$

(2) For every open set $G \in \left(\mathbb{R}^{\infty}, \rho_{\infty}\right)$

$$\underline{\lim}_{\varepsilon\to 0} q(\varepsilon)\log P\left(\left(X_n^{(\varepsilon)}\right)_{n\ge m} \in G\right) \ge -\inf_{\underline{u}\in G} J(\underline{u}).$$

As was mentioned in Introduction, the LDP for $\left\{X_n^{(\varepsilon)}, n \ge m\right\}$ is implied by the LDP for family $\varepsilon\xi$ (ξ is a copy of ξ_n).

In the work of F.K. Klebaner and R.Sh. Liptser [1] found sufficient conditions for the existence of a LDP for the family ξ, and these conditions are formulated only in terms of the distribution . It is assumed that (see [1])

(C_1)

- $E\xi = 0$

- $Ee^{t\xi} < \infty,\ t \in R$ "Cramer's condition".

(C_2) With a cumulant function $H(t) = \log Ee^{t\xi}$ and the Fenchel-Legendre $L(v) = \sup_{t\in R}\left[tv - H(t)\right]$, there exist a function $q(\varepsilon)$, decreasing to 0 as $\varepsilon \downarrow 0$, and a nonnegative function $I(v) = \lim_{\varepsilon\to 0} q(\varepsilon) L(v/\varepsilon), v \in R$ with properties:

- $I(0) = 0$

- $\lim_{|v|\to\infty} I(v) = \infty$

(C_3) If $I(v) < \infty$ for some v, then $t_v^{\varepsilon} = \arg\max\left(t\frac{v}{\varepsilon} - H(t)\right)$ is finite and

$$\overline{\lim_{\varepsilon\to 0}} \frac{q(\varepsilon)}{\varepsilon}\left|t_v^{\varepsilon}\right| < \infty \text{ va } \lim_{\varepsilon\to 0} \varepsilon^2 H''\left(t_v^{\varepsilon}\right) = 0$$

We notice also that the Cramer condition implies $H'(0)=0$ and $H''(t)\geq 0$ and left continity of $I(v)$ in a vicinity of $v_0=\inf\{v>0:I(v)=\infty\}$ (correspondingly, right continuity for $v_0<0$).**Klebaner F and Lipster R in [1] proved the following theorem.**

Theorem (see[1]). Assume **(C_1)- (C_3)**. Then:

1) the family $\{\varepsilon\xi\}_{\varepsilon\to 0}$ obeys to the LDP with rate of speed $q(\varepsilon)$ and the function $I(v)$ defined in **(C_2);**

2) the family $\left\{\{\varepsilon\xi\}_{k\geq 1}\right\}_{\varepsilon\to 0}$ obeys the LDP in the metric space $\left(R^\infty,\rho\right)$ with the rate function $\left(\underline{v}=(v_1,v_2,...)\in R^\infty\right)$.

$$I_\infty(\underline{v})=\sum_{k=1}^{\infty} I(v_k);$$

3) the family $\left\{\left(X_k^\varepsilon\right)_{k\geq m}\right\}_{\varepsilon\to 0}$ obeys the LDP in the metric space $\left(R^\infty,\rho\right)$ with the rate function $\left(\underline{u}=(u_m,u_{m+1},...)\in R^\infty\right)$

$$J_\infty(\underline{u})=\begin{cases}\sum\limits_{k=m}^{\infty}\inf\limits_{v_k:u_k=f(u_{k-1},...u_{k-m},v_k)} I(v_k), & u_i=x_i, i=0,...,m-1\\ \infty, & otherwise,\end{cases}$$

where $\inf(\varnothing)=\infty$.

3. Proof of the Theorem 1.1

In the section we prove the Theorem 1.1.

Prove. By definition

$$f(x)=a\left|\sin\frac{\pi}{2}x\right|,\ 0<a<1,\ x\in[-1,1].$$

We find the infimum of the following rate function

$$I(y_0,y_1,...,y_N)=\frac{1}{2}\sum_{n=1}^{N}\left(y_n-f(y_{n-1})\right)^2=\frac{1}{2}\sum_{n=1}^{N}\left(y_n-a\sin\left(\frac{\pi}{2}y_{n-1}\right)\right)^2$$

We consider the case $N=2$. Then $S(y_1)=y_1^2+\left(1-f(y_1)\right)^2=y_1^2+\left(1-a\sin\left(\frac{\pi}{2}y_1\right)\right)^2$

It is easy to see that

$$S'(y_1)=2y_1+2\left(1-a\sin\left(\frac{\pi}{2}y_1\right)\right)\left(-a\cos\left(\frac{\pi}{2}y_1\right)\right)\frac{\pi}{2}=$$

$$=2y_1-a\pi\cos\left(\frac{\pi}{2}y_1\right)\left(1-a\sin\left(\frac{\pi}{2}y_1\right)\right)=2y_1-a\pi\cos\left(\frac{\pi}{2}y_1\right)+a^2\frac{\pi}{2}\sin(\pi y_1)$$

We have

$S(0)=1$, $S(1)=1+(1-a)^2$, $S'(0)=-a\pi$, $S'(1)=2$ (1.4)

We find the second derivative $S''(y_1)$:

$$S''_a(y_1)=\left[2y_1-a\pi\cos\left(\frac{\pi}{2}y_1\right)+a^2\frac{\pi}{2}\sin(\pi y_1)\right]'=$$

$$=2+a\frac{\pi^2}{2}\sin\left(\frac{\pi}{2}y_1\right)+a^2\frac{\pi^2}{2}\cos(\pi y_1)=2+a\frac{\pi^2}{2}\left(\sin\left(\frac{\pi}{2}y_1\right)+a\cos(\pi y_1)\right)=$$

$$=2+a\frac{\pi^2}{2}\left(\sin\left(\frac{\pi}{2}y_1\right)+a\left(\cos^2\left(\frac{\pi}{2}y_1\right)-\sin^2\left(\frac{\pi}{2}y_1\right)\right)\right)=$$

$$=2+a\frac{\pi^2}{2}\left(\sin\left(\frac{\pi}{2}y_1\right)-a\sin^2\left(\frac{\pi}{2}y_1\right)+a\cos^2\left(\frac{\pi}{2}y_1\right)\right)$$

Since $0<a<1$ and $0\le y_1\le 1$, we have

$$\sin\left(\frac{\pi}{2}y_1\right)-a\sin^2\left(\frac{\pi}{2}y_1\right)=\sin\left(\frac{\pi}{2}y_1\right)\left(1-a\sin\left(\frac{\pi}{2}y_1\right)\right)\ge 0 ,$$

Consequently,

$S''_a(y_1)\ge 2$, $\forall a\in(0,1)$ and $y_1\in[0,1]$.

The last inequality shows that the function $S_a(y_1)$ is convex up on interval $[0,1]$. This , together with (1.4) i.e. initial values of $S_a(y_1)$ and $S'_a(y_1)$ at baundary points $y_1=0, y_1=1$. We get that there exists a point $\chi_a\in(0,1)$, such that ,

$P_a=S_a(\chi_a)<1$ .The theorem 1.1 completely proved.

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