International Journal of Education, Social Science & Humanities.
Finland Academic Research Science Publishers
ISSN: 2945-4492 (online) | (SJIF) = 7.502 Impact factor
Volume-11| Issue-9| 2023 Published: |22-09-2023|

LOCAL DERIVATIONS OF METABELIAN FILIFORM LIE ALGEBRAS

https://doi.org/10.5281/zenodo.8411439

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Introduction

Let A be an algebra (not necessary associative). Recall that a linear mapping $D: A \to A$ is said to be a derivation, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. A linear mapping Δ is said to be a local derivation, if for every $x \in A$ there exists a derivation D_x on A (depending on x) such that $\Delta(x) = D_x(x)$.

This notion was introduced and investigated independently by R.V. Kadison [11] and D.R. Larson and A.R. Sourour [12]. The above papers gave rise to a series of works devoted to the description of mappings which are close to automorphisms and derivations of C^*-algebras and operator algebras. R.V. Kadison set out a program of study for local maps in [11], suggesting that local derivations could prove useful in building derivations with particular properties. R.V.Kadison proved in [11, Theorem A] that each continuous local derivation of a von Neumann algebra M into a dual Banach M-bimodule is a derivation. This theorem gave way to studies on derivations on C^*-algebras, culminating with a result due to B.E. Johnson, which asserts that every local derivation of a C^*-algebra A into a Banach A-bimodule is automatically continuous, and hence is a derivation [7, Theorem 5.3].

Let us present a list of finite or infinite dimensional algebras for which all local derivations are derivations:

- C^*-algebras, in particular, the algebra $M_n(\mathbb{C})$ of all square matrices of order n over the field of complex numbers [7, 11];
- the complex polynomial algebra $\mathbb{C}[x]$ [11];
- finite dimensional simple Lie algebras over an algebraically closed field of characteristic zero [3];
- Borel subalgebras of finite-dimensional simple Lie algebras [18];
- infinite dimensional Witt algebras over an algebraically closed field of characteristic zero [5];
- Witt algebras over a field of prime characteristic [19];
- solvable Lie algebras of maximal rank [13].

On the other hand, some algebras (in most cases close to nilpotent algebras) admit pure local derivations, that is, local derivations which are not derivations. Below a short list of some classes of algebras which admit pure local derivations:

- the algebra $C(x)$ of rational functions [11];
- finite dimensional filiform Lie algebras [3];
- solvable Leibniz algebras with abelian nilradicals, which have a one dimensionial complementary space [2];
- the algebra of lower triangular $n\times n$-matrices [6];
- real octonion algebra [4].

2. Local derivations of metabelian filiform Lie algebras

In this subsection we obtain a description of the space of all local derivations of rank zero solvable Lie algebras with filiform nilradical, namely with a so-called metabelian filifrom Lie radical.

A non-abelian Lie algebra L is called a metabelian, if $[L,L]$ is abelian, that is, $L^{(2)}=0$.

We shall consider a metabelian filiform Lie algebra L of dimension $n\geq 7$ with a basis $\{e_1,\dots,e_n\}$ such that

$$[e_1,e_i]=e_{i+1}, \qquad 2\leq i\leq n-1, e_2,e_i]=e_{i+2}+e_{i+3},$$
$$3\leq i\leq n-3, e_2,e_{n-2}]=e_n.$$

By [16, Proposition 3.2.5], any derivation D on L has the strictly lower triangular matrix $(d_{i,j})$ such that:

$$d_{2,1}=0,$$
$$d_{i+1,i}=d_{3,2}, \qquad 3\leq i<n;$$
$$d_{ij}=d_{i-j+2,2}-d_{i-j+1,1}-d_{i-j,1} \qquad 3\leq j<i-1<n.$$

Note that the numbers $d_{i,j}$ $(3\leq i,j\leq n)$ completely determined by $d_{k,1},d_{k,2}$ ($k=3,\dots,n$) and the space of all derivations $\mathrm{Der}(L)$ has the dimension $2n-4$.

Theorem. *A linear mapping Δ on L is a local derivation if and only if it has a strictly lower triangular matrix $(\delta_{i,j})$ with $\delta_{2,1}=0$.*

Proof. It is clear that any local derivation Δ on L has a strictly lower triangular matrix $(\delta_{i,j})$ with $\delta_{2,1}=0$.

Let Δ be a such matrix. Take an arbitrary element $x=\sum_{i=1}^{n}\xi_i e_i$. Let $\Delta(x)=\sum_{i=3}^{n}\zeta_i e_i$. We need to find a derivation D_x such that $\Delta(x)=D_x(x)$.

We shall consider the following possible two cases.

Case 1. Let $\xi_1\neq 0$. Take

$$d_{3,1}=\frac{\zeta_3}{\xi_1},$$

$$d_{4,1}=\frac{\zeta_4}{\xi_1}, \qquad (1)$$

$$d_{i,2}=0, \qquad i=3,\dots,n.$$

Further, for $i=5,\dots,n$ the numbers $d_{i,1}$ define as follows:

$$d_{i,1}=\frac{1}{\xi_1}\left(\zeta_i-\sum_{j=3}^{i-1}d_{i,j}\xi_j\right), \qquad (2)$$

where $d_{ij}=-d_{i-j+1,1}-d_{i-j,1}, 3\leq j<i-1<n$. Then

$$D_x(x)=\sum_{i=3}^{n}\sum_{j=1}^{i-1}d_{i,j}\xi_j e_i=d_{3,1}\xi_1 e_3+d_{4,1}\xi_1 e_4+\sum_{i=5}^{n}d_{i,1}\xi_1 e_i+\sum_{i=5}^{n}\sum_{j=3}^{i-1}d_{i,j}\xi_j e_i$$

$$\overset{(1),(1)(2)}{=}\zeta_3 e_3+\zeta_4 e_4+\sum_{i=5}^{n}\zeta_i e_i-\sum_{i=5}^{n}\sum_{j=3}^{i-1}d_{i,j}\xi_j e_i+\sum_{i=5}^{n}\sum_{j=3}^{i-1}d_{i,j}\xi_j e_i$$

$$=\sum_{i=3}^{n}\zeta_i e_i=\Delta(x).$$

Case 2. Let $\xi_1=\dots\xi_{s-1}=0, \xi_s\neq 0$, where $2\leq s\leq n-1$. Then $\Delta(x)=\sum_{i=s+1}^{n}\zeta_i e_i$. Set

$$d_{i,1}=0, \qquad i=3,\dots,n,$$

$$d_{3,2}=d_{i+1,i}=\frac{\zeta_{s+1}}{\xi_s}, \quad i=3,\dots,n-1 \qquad (3)$$

and

$$d_{i-s+2,2} = \frac{1}{\xi_s}\left(\zeta_i - \sum_{j=s+1}^{i-1} d_{i,j}\xi_j\right), \quad i = s+2,\dots,n,$$
$$d_{i-s+2,2} = 0, \qquad i = n+1,\dots,n+s-2, \qquad (4)$$

where

$$d_{ij} = d_{i-j+2,2}, 3 \le j < i-1 < n. \qquad (5)$$

Then

$$D_x(x) = \sum_{i=3}^{n}\sum_{j=1}^{i-1} d_{i,j}\xi_j e_i = \sum_{i=s+1}^{n}\sum_{j=s}^{i-1} d_{i,j}\xi_j e_i$$
$$= d_{s+1,s}\xi_s e_{s+1} + \sum_{i=s+2}^{n} d_{i,s}\xi_s e_i + \sum_{i=s+2}^{n}\sum_{j=s+1}^{i-1} d_{i,j}\xi_j e_i$$
$$\overset{(5)}{=} d_{s+1,s}\xi_s e_{s+1} + \sum_{i=s+2}^{n} d_{i-s+2,2}\xi_s e_i + \sum_{i=s+2}^{n}\sum_{j=s+1}^{i-1} d_{i,j}\xi_j e_i$$
$$\overset{(5)(3),(5)(3)(4)}{=} \zeta_{s+1}e_{s+1} + \sum_{i=s+2}^{n}\zeta_i e_i - \sum_{i=s+2}^{n}\sum_{j=s+1}^{i-1} d_{i,j}\xi_j e_i + \sum_{i=s+2}^{n}\sum_{j=s+1}^{i-1} d_{i,j}\xi_j e_i$$
$$= \sum_{i=s+1}^{n}\zeta_i e_i = \Delta(x).$$

The proof is complete.

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