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CONSTRUCTION OF A MODIFICATION OF THE TRAPEZOIDAL QUADRATURE FORMULA BASED ON THE THIRD-ORDER LOCAL INTERPOLATION SPLINE FUNCTION WITH A DEFECT EQUAL TO TWO

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Abstract:. Two recently introduced quadrature schemes for weakly singular integrals are investigated in the context of boundary integral equations arising in the isogeometric formulation of Galerkin Boundary Element Method (BEM). In the first scheme, the regular part of the integrand is approximated by a suitable quasi–interpolation spline. In the second scheme the regular part is approximated by a product of two spline functions. The two schemes are tested and compared against other standard and novel methods available in literature to evaluate different types of integrals arising in the Galerkin formulation. Numerical tests reveal that under reasonable assumptions the second scheme convergences with the optimal order in the Galerkin method, when performing h-refinement, even with a small amount of quadrature nodes. The quadrature schemes are validated also in numerical examples to solve 2D Laplace problems with Dirichlet boundary conditions.

Keywords: isogeometric analysis, Galerkin boundary element method, quadrature formulae, quasi–interpolation.

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Introduction. Boundary Element Method (BEM) is a numerical technique to transform the differential problem into an integral one, where the unknowns are defined only on the boundary of the computational domain. The main two advantages of the method are the dimension reduction of the problem and the simplicity to treat external problems. As a major drawback, the integral formulation involves Boundary Integral Equations (BIEs), which contain singular kernel functions. Therefore, robust and precise quadrature formulae are necessary to provide an accurate numerical evaluation. The solution of the considered BIE is then obtained by collocation or Galerkin procedures. The isogeometric formulation of boundary element method (IgA-BEM) has been successfully applied to 2D and 3D problems, such as linear elasticity, fracture mechanics, acoustic and Stokes flows. Recently, the IgA paradigm has been combined for the first time to the Symmetric Galerkin Boundary Element Method (IgA-SGBEM), which has revealed to be very effective among BEM schemes. Moreover, the full potential of B-splines over the more common Lagrangian basis has been recently exploited. In this work we frame the two quadrature procedures in a Galerkin IgA-BEM for the 2D Laplace problem with Dirichlet boundary conditions. In particular, the derived quadrature formulae are obtained using a quasi–interpolation (QI) operator, firstly introduced and then applied to construct quadrature rules for regular integrals. The second procedure has been successfully applied in a Galerkin adaptive BEM using hierarchical B-splines. The authors also provide some theoretical results about the convergence order of the quadrature rule, when h-refinement is performed.

Materials. In this paper we experimentally test both procedures for the regular and weakly singular integrals occurring in the Galerkin formulation. We compare the achieved accuracy with other quadratures available in literature and suitable for the evaluation of the assayed boundary integrals; namely the methods. Moreover, we recall some results about perturbed Galerkin BEM to provide an estimate for the asymptotic accuracy of the quadratures required to obtain the optimal order of convergence.

Methods. The aim of this paper is to present a higher order predictor method for the numerical tracing of implicitly defined curves. This higher order predictor is described based upon the clamped cubic spline interpolation function using previously computed points on the curve to compute the coefficients via divided differences. Some applications are made to the numerical integration of closed implicitly defined curves. The line integral is approximated via a Gauss–Legendre quadrature of the interpolating function.

Results. Numerical continuation (Path following) methods have long served as useful numerical tools in modern mathematics. They are techniques for numerically approximating a solution curve *c* which is implicitly defined by an underdetermined system of equations. There are various objectives for which the numerical approximation of *c* can be used.

In the context of numerical continuation methods, one considers curves which are implicitly defined by an underdetermined system of equations

(1)H(u)=0,where H:Rn+1→Rn is a smooth map.

We shall mean that a map is *smooth* if it has as many continuous derivatives as the discussion requires.

Let u0∈Rn+1 be a root of H such that the Jacobian matrix $H'(u_0)$ has maximal rank. Then it follows from the Implicit Function Theorem that the solution set $H^{-1}(0)$ can be locally parametrized about u_0 with respect to some parameter, say s. We thus obtain the solution curve $c(s)$ of the equation $H(u)=0$.

If we take s to be the arclength, we obtain a smooth curve c:I→Rn+1 for some interval I containing zero, such that for all $s \in I$:

(1) $c(0)=u_0$;

(2) $H'(c(s))\dot{c}(s)=0$;

(3) $||\dot{c}(s)||=1$;

(4) detH'(c(s))c.(s)∗>0.

Here and in the following, B∗ denotes the Hermitian transpose of B, $||u||$ the Euclidean norm of u, H' the total derivative (the Jacobian) of H, and $c^{\cdot}$ the derivative of c with respect to arclength.

One of the important concepts which we use hereafter is the *tangent vector* induced by an $n\times(n+1)$ matrix with maximal rank. It is denoted by $t(A)$ and is defined to be the unique vector $t(A)$ in Rn+1 that satisfies the following conditions:

(1) $At(A)=0$;

(2) $||t(A)||=1$;

(3) detAt(A)∗>0.

Since the solution curve c is characterized by the initial value problem

(2)u ·=t(H′(u)),u(0)=u0,it is evident that the numerical methods for solving initial value problems could be used to numerically trace c. However, in general this is not an efficient approach, since it ignores the strong contractive properties which the curve c has relative to corrector steps in view of the fact that it satisfies the equation $H(u)=0$. In fact, a typical path following method consists of a succession of two steps:

Predictor step: An approximate step along the curve, usually in the general direction of the tangent of the curve.

Corrector step: One or more iterative steps for solving $H(u)=0$ which bring the predicted point back to the curve.

It is usual to call such procedures *predictor corrector path following methods*.

Path following methods usually split into two main categories. The first one is to safely follow the curve as fast as possible, until a certain point is reached. In this category we will get fast results with less accuracy. The second category is to approximate the entire solution curve with some given accuracy. Siyyam and Syam, considered the first category by applying the Euler predictor and Gauss–Newton–Corrector to trace an implicitly defined curve. Modified versions of the trapezoidal and Romberg rules were used to approximate line integrals over implicitly defined curves. The predictor was only of local order two. So, all of their numerical integration results were of order two. One may expect to obtain improved efficiency by using higher order predictors, especially when the solution curve needs to be approximated very well at all points.

However, in higher dimensions Newton-type correctors may become expensive and hence in order to reduce the number of corrector steps and to allow larger predictor steps, it may be advantageous to use higher order predictors.

One can use the Newton and the Hermite interpolation techniques as a predictor. These techniques concerned the approximation of a portion of the solution curve by a polynomial. However, the oscillatory nature of high-degree polynomials and the property that a fluctuation over a small portion of the curve can induce large fluctuations over the solution curve restricts their use.

Now, assume that A is an $n\times n$ strictly diagonally dominant and tridiagonal matrix. Then, A is a nonsingular matrix which implies that the linear system of n equations and n unknowns $Ax=b$ has a unique solution. In this case we will use the Crout Factorization Algorithm for tridiagonal linear system.

Since the Crout factorization algorithm requires only $(5n-4)$ multiplications and divisions and $(3n-3)$ additions and subtractions, we will use it in this paper.

Assume that the points $u_0,u_1,\ldots,u_m$ along the solution curve c have already been generated. There are many ways to generate the first m points if u_0 is given. One can use multistep methods or Runge–Kutta methods of high order to be suitable to the order that we use in the interpolation. We cannot use the Euler predictor, because it is of order two and if we were to loose the accuracy at any point, it would effect the entire result. Assume that the corresponding tangents $t_0=t(H'(u_0)),\ldots,t_m=t(H'(u_m))$ are computed. For more details how to compute them.

The idea is to use a cubic spline interpolating function $p_q(h)$ using the points $u_m,u_{m-1},\ldots,u_{m-q}$ where $q\leqslant m$, with coefficients in Rn+1, satisfying $p_q(0)=u_m$ as a predicting function. In this case, we say that $p_q(h)$ has order q. The main issue is to express the interpolating function in terms of a suitable parameter ξ. Lundberg and Poore [4] showed that the arclength is the ideal parameter to use. This will give additional complexity of obtaining precise numerical approximations of the arclength s_i such that $c(s_i)=u_i$. For this reason, we use a local parametrization ξ induced by the current approximate tangent $t\approx t(H'(u_m))$, which does not need to be very accurate. We assume the normalization $||t||=1$ holds. This local parametrization $c(\xi)$ is defined as the locally unique solution of the system(3)H(u)=0,t∗(um+ξt−u)=0for ξ in some open interval containing zero. It follows that(4)c(ξi)=ui,whereξi=t∗(ui−um).

By differentiating $c(\xi)$ with respect to ξ and using (3), we obtaindc(ξ)dξ=ċ(s)t∗ċ(s).

We should mention that we have two different types of derivative for c. The first one is the derivative with respect to the arclength which is denoted by ċ(s). In this paper, we use the first derivative only with respect to the arclength. The second type is the derivative with respect to ξ. The notation for these derivatives are $dc(\xi)/d\xi=c'(\xi)$, $c''(\xi)$, $c^{(3)}(\xi),\ldots,c^{(m)}(\xi)$.

If the tangents t_i at the points u_i are available for use, we can form the clamped cubic spline interpolating function p_q.

The clamped cubic spline function $p_q(h)$ satisfies the following conditions:

(a)

$p_q(h)$ is a cubic polynomial, denoted by $S_j^q(h)$, on the subinterval $[\zeta_j,\zeta_{j+1}]$ for each $j=m-q:m-1$.

(b)

$p_q(\zeta_j)=c(\zeta_j)$ for each $j=m-q:m$.

(c)

$S_{j+1}{}^{q}(\zeta_{j+1})=S_j{}^{q}(\zeta_{j+1})$ for each $j=m-q:m-2$.

(d)

$S_{j+1}{}^{q'}(\zeta_{j+1})=S_j{}^{q'}(\zeta_{j+1})$ for each $j=m-q:m-2$.

(e)

$S_{j+1}{}^{q''}(\zeta_{j+1})=S_j{}^{q''}(\zeta_{j+1})$ for each $j=m-q:m-2$.

(f)

$p_q'(\zeta_{m-q})=c'(\zeta_{m-q})$ and $p_q'(\zeta_m)=c'(\zeta_m)$.

where ξ_i and $c[\xi_i]$ are as given in (4) and c′[ξi]=ti/(t∗ti).

To construct the clamped cubic spline interpolant for $c(\zeta)$, the conditions (a)–(f) are applied to the cubic polynomialsSjq(h)=aj+bj(h−ζj)+cj(h−ζj)2+dj(h−ζj)3for each $j=m-q:m-2$. From condition (b), we see that $p_q(\zeta_j)=S_j{}^{q}(\zeta_j)=c(\zeta_j)$ which implies thataj=c(ζj)=ujforeachj=m−q:m−1.Simple calculations give us the following linear system(5)Ax=b,whereA=2hm−qhm−q0⋯⋯0hm−q2(hm−q+hm−q+1)hm−q+1⋮0hm−q+12(hm−q+1+hm−q+2)⋱⋮⋮⋱⋱⋱⋱⋮⋮⋱⋱⋱0⋮hm−22(hm−2+hm−1)hm−10⋯⋯0hm−12hm−1,b=3hm−q(am−q+1−am−q)−3c′(ζm−q)3hm−q+1(am−q+2−am−q+1)−3hm−q(am−q+1−am−q)⋮3hm−1(am−am−1)−3hm−2(am−1−am−2)3c′(ζm)−3hm−1(am−am−1)andx=cm−qcm−q+1cm−q+2⋮cm.

In the linear system (5), $h_j=\zeta_{j+1}-\zeta_j$, for each $j=m-q:m-1$.

Since A is strictly diagonally dominant, the linear system has a unique solution for c_{m-q}, $c_{m-q+1},\ldots,c_m$. From condition (e), we see that2cj+1=2cj+6djhjwhich implies thatdj=cj+1−cj3hjfore chj=m−q:m−1.From condition (c), we have $a_{j+1}=a_j+b_jh_j+c_jh_j^2+d_jh_j^3$ which implies thatbj=aj+1−ajhj−cjhj−djhj2=aj+1−ajhj−hj3(2cj+cj+1)for each $j=m-q:m-1$.

We should note that the linear system is tridiagonal system and A is strictly diagonally dominant. Thus, we will use the Crout factorization for solving this system.

Now a general philosophy for monitoring the order and steplength of higher order predictors is presented. To do this, let u_n be a current point on the solution curve c which can be locally parametrized via the parameter s, and assume that $c(0)=u_n$. For a cubic spline of order q, consider a polynomial predictor of the formc(h)≈Sjq(h)=uj+∑i=14ci,jhi,ξj⩽h⩽ξj+1,(6)ci,j≈c(i)(ξj)i!which represents an approximation via the Taylor formula. For more details, how can we write. In fact, there are two different ways for obtaining the coefficients $c_{i,j}$.

(1) By polynomial interpolation making use of previously calculated points on the curve.

(2) By successive numerical differentiation at u_n.

The former is less expensive to calculate, and it is the approach which will be presented in this paper.

One way for determining the next steplength and the next order in the predictor is given below. Let tol>0 be a given tolerance. The term $||c_{4,j}||h^4$ can be viewed as a rough estimate for the truncation error of the predictor $p_{q-1}(h)$ in the interval $[\xi_j,\xi_{j+1}]$. Hence, by solving $||c_{4,j}||h^4$=tol for h, we get(7)hj(q)=tol||c4,j||1/4.Let $h_q=\max\{h_j^{(q)}:m-q\leqslant j\leqslant m-1\}$. Choose h_q as the steplength for the predictor $p_{q-1}(h)$ in order to remain within the given tolerance. Due to instabilities of various kinds, we anticipate that eventually(8)h2<h3<⋯<hl⩾hl+1,will hold for some l. Thus, the predictor p_{l-1} with steplength h_l is our next choice.

One of the interesting cases for the line integral is the integration over a closed curve. In order to handle this case for an implicitly defined curve, it is necessary to develop a reliable numerical method for determining when the curve has been completely traversed. Siyyam and Syam developed such stopping criteria. We have implemented that stopping criterion along with the clamped cubic spline higher order predictor and tested many different examples. The results we have obtained indicate that it works nicely and efficiently.

In this example we took the maximum degree of any clamped cubic spline interpolating function to be eight. Moreover, we have calculated the sum of all stepsizes from the steplength control $h_1, h_2,\ldots,h_{n-1}$ where n is the number of points generated along the solution curve until the stopping criterion is satisfied, along with the corresponding predictor's degrees $q_1, q_2,\ldots,q_{n-1}$, and we define S_k to be Sk=∑i=1khi for k=1:n−1. We have sketched the graphs of h_k against S_k, and the graphs of q_k against S_k for k=1:n−1 for the tolerances 10^{-6}, 10^{-9}, and 10^{-12}, respectively.

In Example (1), we choose a symmetric curve which is an ∞-Shape. If the arclength control (8) is satisfied at all points along the solution curve, two conditions must hold.

Conclusion. A study of the two recently introduced spline quasi–interpolation quadrature schemes is performed in the context of boundary integral equations in Galerkin IgA-BEM. A comparison of the accuracy of the schemes was already done, when considering singular integrals. The analysis with respect to the amount of employed quadrature nodes revealed the optimal order of convergence for both approaches. In the present paper, numerical tests show a notable difference between the two schemes. For a fixed amount of quadrature nodes the accuracy of the considered integrals is examined, when performing h-refinement of the approximation space. The observed rate of convergence is optimal only for the second scheme. In the numerical simulations for the 2D Laplace problems, the optimal order of convergence of the approximate solution is achieved with a small number of quadrature nodes, when the second procedure is employed. Regarding the first procedure, the amount of nodes should be increased to recover the optimal order for all the h-refinement steps. In the future work we would like to investigate quadrature schemes for integrals of higher order singularities for more complex differential problems. A valuable contribution would be to derive stable formulae for the modified moments to simplify the construction of the proposed methods.

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