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O‘ZGARUVCHI TARTIBLI KAPUTO-FABRITSIO OPERATORI ISHTIROK ETGAN DIFFERENSIAL TENGLAMA UCHUN CHEGARAVIY MASALA

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Annotatsiya.

Ushbu ishda vaqt o‘zgaruvchisi bo‘yicha o‘zgaruvchi tartibli Kaputo-Fabritsio operatori, fazoviy o‘zgaruvchi bo`yicha esa Lejandr operatori qatnashgan xususiy hosilali differensial tenglama uchun lokal shartli masalaning bir qiymatli yechilishi isbotlangan. Bunda Lejandr polinomlarining xossalaridan foydalanilgan.

Kalit so‘zlar.

O‘zgaruvchi tartibli Kaputo-Fabritsio operatori; Lejandr polinomlari; Furye usuli.

BOUNDARY PROBLEM DIFFERENTIAL EQUATION WITH CAPUTO-FABRIZIO OPERATOR OF VARIABLE ORDER

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Abstract.

In this work a unique solvability of nonlocal boundary problem for partial differential equation involving Caputo-Fabrizio operator of variable order in time and Legendre operator in space-variable. Properties of Legendre polynomials have been used.

Keywords.

Caputo-Fabrizio operator; Legendre polynomials; Fourier method.

Ushbu ishda o‘zgaruvchi tartibli Kaputo-Fabrizio operatori qatnashgan to‘lqin tenglamasi uchun bir chegaraviy masalaning bir qiymatli yechilishini tadqiq etamiz.

Masalaning qo‘yilishi. Shunday $U(x,t)\in C\left(\bar{\Omega}\right)\cap C^2\left(\Omega\right)$ funksiya topilsinki, u $\Omega=\left\{(x,t):-1<x<1,0<t<T\right\}$ sohada

$$ {}_{C}^{CF}D_{0t}^{\alpha(t)}U=[(1-x^2)U_x]_x+f(x,t) \quad (1)$$

tenglamani va

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$$U(x,0)=\varphi(x),\ \ -1\le x\le 1 \qquad U_t(x;0)=\psi(x),\ \ -1<x<1 \qquad (2)$$

boshlang‘ich shartlarni qanoatlantiruvchi, hamda

$x=-1,\ x=1$ da $U(x,t), U_x(x,t)$ chegaralangan bo‘lsin.

Bu yerda

$$ {}_{C}^{CF}D_{0t}^{\alpha(t)}U=\frac{B(\alpha(t)-1)}{2-\alpha(t)}\int_0^t e^{\frac{\alpha(t)-1}{2-\alpha(t)}(t-s)}U''(s)ds\ , \qquad 1<\alpha(t)<2 \qquad (3)$$

$\alpha(t)$ o‘zgaruvchi kasr tartibli Kaputo-Fabritsio operatori [1], $B(\alpha(t))$ normallashtiruvchi funksiya bo‘lib, $B(0)=B(1)=1$ shartni qanoatlantiradi. $f(x,t)$, $\varphi(x),\psi(x)$ funksiyalar esa berilgan funksiyalar bo’lib, $\varphi(0)=0,\ \varphi(1)=0$.

Masalaning yechilishi. Dastlab $f(x,t)\equiv 0$ bo‘lgan holda (1) tenglamaning yechimini $U(x,t)=\upsilon(t)\cdot X(x)\neq 0$ ko‘rinishda qidiramiz. U holda (1) dan

$$ {}^{CF}D_{0t}^{\alpha(t)}\upsilon(t)\cdot X(x)=[(1-x^2)X''(x)-2xX'(x)]\upsilon(t)$$

ni olamiz. Bu tenglikni $\upsilon(t)\cdot X(x)$ ga bo‘lsak

$$\frac{{}^{CF}D_{0t}^{\alpha(t)}\upsilon(t)}{\upsilon(t)}=\frac{[(1-x^2)X''(x)-2xX'(x)]}{X(x)}=-\lambda.$$

yoki

$$ {}^{CF}D_{0t}^{\alpha(t)}\upsilon(t)+\lambda\upsilon(t)=0, \qquad (4)$$

$$(1-x^2)X''(x)-2xX'(x)+\lambda X(x)=0 \qquad (5)$$

tenglamalar kelib chiqadi. $U(x,t), U_x(x,t)$ funksiyalarning $x=-1,\ x=1$ da chegaralangan yechimga ega ekanligi ma’lum hamda bu yechim

$$X(x)=P_n(x)=\frac{1}{2^n\cdot n!}\cdot\frac{d^n(x^2-1)^n}{dx^n}\quad (n=0,1,2,...) \qquad (6)$$

Lejandr polinomlari orqali ifodalanadi [2].

Ma’lumki [2], $P_n(x)$ $(n=0,1,2,...)$ Lejandr polinomlari [-1;1] da ortogonal sistema tashkil qiladi va $\| P_n(x)\|^2=\frac{2}{2n+1}$.

Ihtiyoriy [-1;1] da bo‘lakli uzlukli $g(x)$ funksiyani $\{P_n(x)\}$ sistema bo‘yicha Furye qatoriga yoyish mumkin [2]:

$$g(x)=\sum_{n=0}^{\infty}C_nP_n(x)\ ,\ C_n=\frac{(g,P_n)}{\| P_n\|^2}=\frac{2n+1}{2}\int_{-1}^{1}g(x)P_n(x)dx.$$

Bunday qatorlarni Furye-Lejandr qatorlari deyiladi.

Masala yechimini Furye-Lejandr qatori ko‘rinishida qidiramiz:

$$u(x,t)=\sum_{n=0}^{\infty}V_n(t)P_n(x), \qquad (7)$$

bu yerda $V_n(t)$ hozircha noma'lum funksiyalar.

Berilgan funksiyalar $f(x,t),\varphi(x),\psi(x)$ larni ham Furye-Lejandr qatoriga yoyamiz:

$$f(x,t)=\sum_{n=0}^{\infty}f_n(t)P_n(x), \qquad (8)$$

$$\varphi(x)=\sum_{n=0}^{\infty}\varphi_nP_n(x), \qquad \psi(x)=\sum_{n=0}^{\infty}\psi_nP_n(x), \qquad (9)$$

bu yerda

$$f_n(t)=\frac{2n+1}{2}\int_{-1}^{1}f(x,t)P_n(x)dx, \quad \varphi_n=\frac{2n+1}{2}\int_{-1}^{1}\varphi(x)P_n(x)dx,$$

$$\psi_n=\frac{2n+1}{2}\int_{-1}^{1}\psi(x)P_n(x)dx.$$

(8) va (9) ni (1) va (2) ga qo'yish natijasida quyidagi Koshi masalasiga ega bo'lamiz:

$$ {}_{C}^{CF}D_{0t}^{\alpha(t)}V_n(t)+n(n+1)V_n(t)=f_n(t), \qquad (10)$$

$$V_n(0)=\varphi_n, \qquad V_n'(0)=\psi_n. \qquad (11)$$

Bu yerda ψ_n va φ_n lar mos ravishda $\psi(x)$ va $\varphi(x)$ funksiyalarning Furye-Lejandr koeffitsiyentlari.

(10) tenglamaning operator qatnashgan qismini soddalashtiramiz.

$$ {}_{C}^{CF}D_{0t}^{\alpha(t)}V_n(t)=\frac{B(\alpha(t)-1)}{2-\alpha(t)}\int_0^t e^{\frac{\alpha(t)-1}{2-\alpha(t)}(t-s)}V_n''(s)ds=$$

$$=\frac{B(\alpha(t)-1)}{2-\alpha(t)}e^{\frac{\alpha(t)-1}{2-\alpha(t)}t}\int_0^t e^{\frac{1-\alpha(t)}{2-\alpha(t)}s}V_n''(s)ds.$$

$M(\alpha(t))=\dfrac{\alpha(t)-1}{2-\alpha(t)}$ belgilash kiritsak, oxirgi tenglik ushbu ko'rinishni oladi:

$$ {}_{C}^{CF}D_{0t}^{\alpha(t)}V_n(t)=\frac{B(\alpha(t)-1)}{2-\alpha(t)}e^{M(\alpha(t))t}\int_0^t e^{-M(\alpha(t))s}V_n''(s)ds.$$

Endi bu tenglikdagi integralni soddalashtiramiz. Bo'laklab integrallash qoidasini qo'llasak,

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$$\int_0^t e^{-M(\alpha(t))s} V_k''(s)ds = V_n'(s)e^{-M(\alpha(t))s}\Big|_0^t + \int_0^t M(\alpha(t))e^{-M(\alpha(t))s} V_n'(s)ds =$$

$$= V_n'(t)e^{-M(\alpha(t))} - V_n'(0) + \int_0^t M(\alpha(t))e^{-M(\alpha(t))s} V_n'(s)ds =$$

$$= V_n'(t)e^{-M(\alpha(t))} - \psi_n + \int_0^t M(\alpha(t))e^{-M(\alpha(t))s} V_n'(s)ds$$

tenglik hosil bo'ladi.

U holda

$$\frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} \int_0^t e^{-M(\alpha(t))s} V_n''(s)ds =$$

$$= \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} \left(V_n'(t)e^{-M(\alpha(t))} - \psi_n + \int_0^t M(\alpha(t))e^{-M(\alpha(t))s} V_n'(s)ds \right) =$$

$$= \frac{B(\alpha(t)-1)}{2-\alpha(t)} V_n'(t) - \psi_n \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} +$$

$$+ \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} \int_0^t M(\alpha(t))e^{-M(\alpha(t))s} V_n'(s)ds =$$

$$= \frac{B(\alpha(t)-1)}{2-\alpha(t)} V_n'(t) - \psi_n \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} +$$

$$+ \int_0^t M(\alpha(t)) \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))(t-s)} V_n'(s)ds\,;$$

Tenglamada qatnashgan $n(n+1)V_n(t)$ ifodani quyidagi ko'rinishda yozib olamiz:

$$n(n+1)V_k(t) = n(n+1)\int_0^t V_n'(s)ds + n(n+1)V_n(0)$$

Hosil qilingan tengliklarni (10) tenglamaga olib borib qo'yamiz. Natijada tenglama quyidagicha ko'rinishga keladi:

$$\frac{B(\alpha(t)-1)}{2-\alpha(t)} V_n'(t) - \psi_n \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} +$$

$$+ \int_0^t M(\alpha(t)) \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))(t-s)} V_n'(s)ds + n(n+1)\int_0^t V_n'(s)ds + n(n+1)\varphi_n = f_n(t)$$

$$\frac{B(\alpha(t)-1)}{2-\alpha(t)} V_n'(t) + \int_0^t \left(\frac{M(\alpha(t))B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))(t-s)} + n(n+1) \right) V_n'(s)ds =$$

$$= f_n(t) - n(n+1)\varphi_n + \psi_n \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t}$$

$$V_n'(t) + \int_0^t \left(M(\alpha(t)) e^{M(\alpha(t))(t-s)} + \frac{2-\alpha(t)}{B(\alpha(t)-1)} n(n+1) \right) V_n'(s)ds =$$

$$= \frac{2-\alpha(t)}{B(\alpha(t)-1)} \left(f_n(t) - n(n+1)\varphi_n + \psi_n \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} \right)$$

hosil bo‘lgan tenglikdan

$$\tilde{f}_n(t) = \frac{2-\alpha(t)}{B(\alpha(t)-1)} \left(f_n(t) - n(n+1)\varphi_n + \psi_n \frac{B(\alpha(t)-1)}{2-\alpha(t)} e^{M(\alpha(t))t} \right)$$

$$K(t,s) = M(\alpha(t)) e^{M(\alpha(t))(t-s)} + \frac{2-\alpha(t)}{B(\alpha(t)-1)} n(n+1)$$

kabi belgilash kiritsak, (10) tenglama quyidagi ikkinchi tur Volterra integral tenglamasiga keladi:

$$V_n'(t) + \int_0^t K(t,s) V_n'(s)ds = \tilde{f}_n(t) \qquad (12)$$

yoki

$$V_n'(t) = \tilde{f}_n(t) - \int_0^t K(t,s) V_n'(s)ds$$

$K(t,s)$ uzluksiz va $\tilde{f}_k(t)$ uzluksiz differensiallanuvchi bo‘lsa, (12) tenglama yagona yechimga ega bo‘ladi.

Endi hosil bo‘lgan Volterra integral tenglamasini tadqiq etamiz. Biz integral tenglamaning rezolventasini topib, $V_n'(t)$ noma‘lum funksiyaning yaqqol ko‘rinishini topamiz. (12) tenglamani ketma-ket yaqinlashish usuli yordamida ishlashga kirishamiz.

Nolinchi yaqinlashish sifatida $\tilde{f}(t)$ ni qabul qilamiz: $\left(V_n'(t) = \tilde{V}(t)\right)$

$\tilde{V}_0(t) = \tilde{f}(t)$.

$\tilde{V}_1(t)$ ni esa

$$\tilde{V}_1(t) = \tilde{f}(t) - \int_0^t K(t,s) \tilde{f}(s)ds$$

munosabat bilan aniqlaymiz. Bu jarayonni davom ettirib,

$$\tilde{V}_n(t) = \tilde{f}(t) - \int_0^t K(t,s) \tilde{V}_{n-1}(s)ds$$

rekkurent formulani qanoatlantiruvchi

$\tilde{V}_0(t), \tilde{V}_1(t), \ldots\ldots, \tilde{V}_n(t), \ldots.$

funksiyalarning cheksiz ketma ketligini hosil qilamiz.

Ikkinchi yaqinlashish

$$\tilde{V}_2(t) = \tilde{f}(t) - \int_0^t K(t,s)\tilde{V}_1(s)ds$$

formula bilan aniqlangani uchun, $\tilde{V}_1(s)$ ning o‘rniga uning qiymatini qo‘yib, Dirixle formulasidan foydalansak,

$$\tilde{V}_2(t) = \tilde{f}(t) - \int_0^t K(t,s)\left[\tilde{f}(s) - \int_0^s K(s,z)\tilde{f}(z)dz\right]ds =$$

$$= \tilde{f}(t) - \int_0^t K(t,s)\tilde{f}(s)ds + \int_0^t K(t,s)\left[\int_0^s K(s,z)\tilde{f}(z)dz\right]ds =$$

$$= \tilde{f}(t) - \int_0^t K(t,s)\tilde{f}(s)ds + \int_0^t \tilde{f}(z)dz\int_z^t K(t,s)K(s,z)ds$$

tenglik hosil bo‘ladi,bu yerda

$$K_2(t,z) = \int_z^t K(t,s)K(s,z)ds.$$

Xuddi shu yo‘l bilan $\tilde{V}_n(t)$ uchun

$$\tilde{V}_n(t) = \tilde{f}(t) - \int_0^t \sum_{i=1}^n K(t,s)K_{i-1}(s,z)ds$$

formula hosil qilamiz, bunda

$$K_i(t,z) = \int_z^t K(t,s)K_{i-1}(s,z)ds.$$

Bizning tenglamda

$$K(t,s) = \frac{\alpha(t)-1}{2-\alpha(t)}e^{\frac{\alpha(t)-1}{2-\alpha(t)}(t-s)} + \frac{2-\alpha(t)}{B(\alpha(t)-1)}n(n+1).$$

Shu sababli yuqoridagi formuladan foydalansak, ikkinchi yaqinlashish quyidagi ko‘rinishga ega bo‘ladi:

$$K_2(t,z)=\int_z^t\left(\frac{\alpha(t)-1}{2-\alpha(t)}e^{\frac{\alpha(t)-1}{2-\alpha(t)}(t-s)}+\frac{2-\alpha(t)}{B(\alpha(t)-1)}n(n+1)\right)\times$$
$$\left(\frac{\alpha(s)-1}{2-\alpha(s)}e^{\frac{\alpha(s)-1}{2-\alpha(s)}(s-z)}+\frac{2-\alpha(s)}{B(\alpha(s)-1)}n(n+1)\right)ds$$

Xuddi shu tartibda davom etib, $V_n'(t)$ ni umumiy koʻrinishini rezolventa orqali ifodalab quyidagicha yozib olamiz:

$$V_n'(t)=\tilde{f}_n(t)-\int_0^t R(t,s)\,\tilde{f}_n(s)ds$$

Bu tenglikning har ikki tomonini integrallab, $V_n(t)$ ni yaqqol koʻrinishini quyidagicha topamiz:

$$V_n(t)=\int_0^t \tilde{f}_n(s)ds-\int_0^t\left(\int_0^s R(z,s)\,\tilde{f}_n(z)dz\right)ds+\varphi_n\,. \qquad (13)$$

(13) ifodani (8) tengliklarga olib borib qoʻyib, cheksiz qatorga ega boʻlamiz. Berilgan funksiyalarga maʻlum shatrlar asosida cheksiz qatorlarning tekis yaqinlashishini isbotlaymiz.

Bizda quyidagi qator hosil boʻladi.

$$U(x;t)=\sum_{n=0}^{\infty}\left(\int_0^t \tilde{f}_n(s)ds-\int_0^t\left(\int_0^s R(z,s)\,\tilde{f}_n(z)dz\right)ds+\varphi_n\right)P_n(x). \qquad (14)$$

$U(x;t)$ qatorni bahosini olamiz. Buning uchun $|U(x;t)|\le\sum_{n=0}^{\infty}|V_n(t)|$

tengsizlik bajarilishi kerak. Tengsizlikning oʻng tomonida qatnashgan qatorni yaqinlashuvchi boʻlishi uchun $V_n(t)$ da qatnashgan funksiyalarga ma’lum shatrlar tushadi. Buning uchun $V_1(t)$ ning ifodasidan foydalanamiz [3]:

$$V_1(t)=\int_0^t \tilde{f}_1(s)ds-\int_0^t\left(\int_0^s R(z,s)\,\tilde{f}_1(z)dz\right)ds+\varphi_1$$

$\sum_{n=0}^{\infty}\frac{C_1}{n(n+1)}$ qator yaqinlashuvchi boʻlganli sababli,

$$|U(x;t)|\le\sum_{n=0}^{\infty}|V_n(t)|\le\sum_{n=0}^{\infty}\frac{C_1}{n(n+1)}$$

yuqoridagi tengsizlik bajarilishini ta’minlasak (14) qatorni ham yaqinlashishini isbotlagan boʻlamiz. Buning uchun (14) qatorning tarkibida ishtirok etayotgan $f_n(t)$ funksiya va φ_n, ψ_n Furye koeffitsiyentlarini bo'laklab integrallab, ko`rsatish mumkinki, ushbu

$$|U(x;t)| \le \sum_{k=0}^{\infty} |V_n(t)| \le \sum_{n=0}^{\infty} \frac{C_1}{n(n+1)} + \sum_{n=0}^{\infty} C_2 \left|\varphi_n^{(5)}\right|^2 + \sum_{n=0}^{\infty} C_3 \left|\psi_n^{(3)}\right|^2 + \sum_{n=0}^{\infty} C_4 \left|f_n^{(3)}\right|^2$$

tengsizlikning o'ng tomonidagi qatorlar yaqinlashuvchi bo'ladi.

Tenglamada ishtirok etadigan $u_x, u_{xx}, {}^{CF}D_{0t}^{\alpha}u$ funksiyalarga mos keluvchi cheksiz qatorlarning tekis yaqinlashishini isbotlashda berilgan $\varphi(x), f(x,t)$ funksiyalarga ko'proq shart tushishi tabiiy. Bunda [4] da ko'rsatilganidek hisoblashlar asosida $|g_n| \le \frac{4\sqrt{2}}{(2n-3)^{3/2}} \|g\|$ tengsizliklardan foydalanamiz.

Masala yechimi yagonaligi esa Furye-Lejandr qatorlari nazariyasiga ko'ra Lejandr polinomlarining to'la ortonormal sistema tashkil qilishidan foydalanib isbotlanadi. Demak, quyidagi tasdiq o'rinli:

Teorema. Agar $\varphi(x) \in C^4[-1,1]$, $\varphi^{(5)} \in L_2(-1,1)$, $\varphi(-1) = \varphi(1) = 0$, $\varphi''(-1) = \varphi''(1) = 0$, $\varphi^4(-1) = \varphi^4(1) = 0$, $\psi \in C^2[-1,1]$, $\psi''' \in L_2(-1,1)$, $\psi(-1) = \psi(1) = 0$, $\psi''(-1) = \psi''(1) = 0$, $f(\cdot,t) \in C^2[-1,1]$, $\frac{\partial^3 f}{\partial x^3}(\cdot,t) \in L_2(-1,1)$ shartlar bajarilsa, qo'yilgan masalaning yechimi mavjud va yagona bo'ladi.

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