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CHEKLI O`LCHAMLI SIMPLEKSDA ANIQLANGAN BISTOXASTIK OPERATORLARNING BA`ZI XOSSALARI HAQIDA

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Annotatsiya

Ushbu ishda chekli o'lchamli simleksda aniqlangan qat'iy musbat bistoxastik operatorlarning qo'zg'almas nuqtalari va ularning tipi o'rganilgan. Qat'iy musbat bistoxastik operatorlarning trayektoriyalari haqida teoremalar isbotlangan.

Kalit so'zlar

Qo'zg'almas nuqta, stoxastik operator, bistoxastik operator, simpleks.

Аннотация

В данной работе изучаются неподвижные точки и их типы положительно определенных бистохастических операторов, определенных в конечномерном симплексе. Доказаны теоремы о траекториях бистохастических операторов.

Ключевые слова

Неподвижная точка, стохастический оператор, бистохастический оператор, симплекс.

Abstract

In this work, fixed points and their type of fixed positive bistochastic operators defined in a finite-dimensional simlex are studied. Theorems about trajectories of positive definite bistochastic operators are proved.

Keywords

Fixed point, stochastic operator, bistochastic operator, simplex.

1.KIRISH

Nochiziqli tenglamalar zamonaviy matematik fizika, genetika va biologiyaning ko`plab masalalarida qo`llaniladi. O'z navbatida bunday masalalar uchun nochiziqli operatorlarning qo`zg`almas nuqtalari va ularning traektoriyasi alohida o`rin egallaydi. Hozirgi kunda katta qiziqish bilan o'rganilib kelinayotgan yo’nalishlardan biri bu, bevosita genetika masalalariga aloqador bo’lgan chekli o’lchamli simpleksdagi nochiziqli bistoxastik operatorlarning dinamikasiga doir masalalar hisoblanadi. Chekli o’lchamli simpleksda aniqlangan bistoxastik operatorlarning qo`zg`almas nuqtalarining xususiyatlarini o'rganish bo'yicha ko'plab nashrlar mavjud [3]-[5], [8], [11]. Ushbu ish chekli o‘lchovli simpleksda bistoxastik operatorlarning traektoriyasini o‘rganishga bag‘ishlangan. Bistoxastik operatorlarning qo‘zg‘almas nuqtalari tipi to‘g‘risida teoremalar isbotlangan.

2.ASOSIY TA`RIF VA TUSHUNCHALAR

Bizga $N_{\leq q}=\{1,2,3,\dots,q\}\subset\mathbb{N}$ (bu yerda $q\in\mathbb{N}$) to’plam va berilgan bo’lsin.

$$S^{m-1}=\{x=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m, \sum_{j=1}^{m}x_j=1,\ x_j\geq 0,\ j\in N_{\leq m}\}$$

to’plamga R^m dagi $m-1$ o’lchamli simleks deyiladi. $S_{>}^{m-1}$ orqali S^{m-1} simpleksning ichki nuqtalari to`plamini belgilaymiz, ya`ni

$$S_{>}^{m-1}=\{x=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m, \sum_{j=1}^{m}x_j=1,\ x_j>0,\ j\in N_{\leq m}\}.$$

S^{m-1} simpleksning chegarasini ∂S^{m-1} orqali belgilaymiz. $e_k=(e_1^{(k)},e_2^{(k)},\dots,e_m^{(k)}),\ (k\in N_{\leq m})$ orqali S^{m-1} simpleksni uchini belgilaymiz, ya`ni

$e_k^{(k)}=1$ va $e_j^{(k)}=0$ barcha $j\neq k,\ k\in N_{\leq m}$ uchun.

Elementlari haqiqiy sonlardan tashkil topgan $m\times m$ kvadrat matritsani $A=(a_{ij})$ orqali belgilaymiz. Agar A - $m\times m$ kvadrat matritsa elementlari uchun $a_{ij}\geq 0,\ \forall i,j\in N_{\leq m}$ va $\sum_{i=1}^{m}a_{ij}=1,\ j\in N_{\leq m}$ tengliklar o`rinli bo`lsa, u holda A stoxastik matritsa deyiladi. S_n orqali $n\in\mathbb{N}$ ta elementning o`rin almashtirishlari guruppasini belgilaymiz.

Tarif 1.1. Ixtiyoriy $\nu\in\mathbb{N}$ uchun, quyidagi operator

$$\mathbf{S}: x=(x_1,x_2,\dots,x_m)\in\mathbb{R}^m\to\varphi(x)=(\varphi_1(x),\varphi_2(x),\dots,\varphi_m(x))\in\mathbb{R}^m \qquad (1.1)$$

ν -tartibli stoxastik operator deyiladi, agar $\mathbf{S}\geq\theta$ va

$$\varphi_k(x)=\sum_{i_1,i_2,\dots,i_\nu=1}^{m}P_{i_1i_2\dots i_\nu,k}x_{i_1}x_{i_2}\dots x_{i_\nu},\ x\in\mathbb{R}^m,\ k\in N_{\leq m} \qquad (1.1.1)$$

bu yerda

$$P_{i_1i_2\dots i_\nu,k}\geq 0,\ i_j=\overline{1,m},\ j=\overline{1,\nu},\ k=\overline{1,m} \qquad (1.1.2)$$

$$P_{i_1i_2\dots i_\nu,k}=P_{i_{\pi(1)}i_{\pi(2)}\dots i_{\pi(\nu)},k},\ k=\overline{1,m} \qquad (1.1.3)$$

ixtiyoriy $\pi\in S_m$ o`rin almashtirish uchun, va

$$\sum_{k=1}^{m}P_{i_1i_2\dots i_\nu,k}=1,\ i_j=\overline{1,m},\ j=\overline{1,\nu}. \qquad (1.1.4)$$

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Yuqoridagi (1.1.2) - (1.1.4) shartlardan

$$\sum_{k=1}^{m} \varphi_k(x) = (x_1 + x_2 + \ldots + x_m)^{\nu}, \; x \in \mathbb{R}^m$$

bo`lishi kelib chiqadi. Ko`rinib turibdiki **S** operator S^{m-1} simpleksni o`zini o`ziga akslantiruvchi operartordir. $\nu = 1$ bo`lganda **S** operator chiziqli stoxastik operator deyiladi, $\nu = 2$ bo`lganda **S** operator kvadratik stoxastik operator deyiladi, $\nu = 3$ bo`lganda **S** operator kubik stoxastik operator deyiladi va hokazo. $\mathbf{S}^{[\nu]}$ orqali ν - tartibli stoxastik operatorni belgilaymiz.

Elementlari haqiqiy sonlardan tashkil topgan $m \times m$ kvadrat matritsani $A = (a_{ij})$ orqali belgilaymiz. Agar A - $m \times m$ kvadrat matritsa elementlari uchun $a_{ij} \geq 0$, $\forall i, j = \overline{1, m}$ $\sum_{i=1}^{m} a_{ij} = 1$, $\forall i, j = \overline{1, m}$ $\sum_{j=1}^{m} a_{ij} = 1$, $\forall i, j = \overline{1, m}$ tengliklar o`rinli bo`lsa, u holda A matritsa bistoxastik matritsa deyiladi. Ba`zan bistoxastik matritsa, $\mathbb{R}^m$dagi bistoxastik operator deb ham ataladi. Ko`rishimiz mumkinki chiziqli bistoxastik operator uchun $A(S^{m-1}) \subset S^{m-1}$ munosabat o`rinli bo`ladi.

Endi umumiy holda bistoxastik operator ta`rifini kiritaylik. Ixtiyoriy $x = (x_1, x_2, \ldots, x_m) \in R^m$ uchun quyidagi $x \downarrow = (x_{[1]}, x_{[2]}, \ldots, x_{[m]})$ ga x ning qayta tartiblanishi deyiladi, bu yerda $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[m]}$.

Tarif 1.2. Faraz qilaylik $x, y \in S^{m-1}$ bo`lsin. Agar barcha $k = \overline{1, m-1}$ uchun $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ tengsizlik o`rinli bo`lsa, u holda $x$ element $y$ ga majorizatsiyalashgan (kattalashgan) deyiladi va $x \prec y$ ko`rinishida belgilanadi.

Tarif 1.3. Agar ν -tartibli stoxastik operator uchun $\mathbf{S}^{[\nu]} x \prec x, \forall x \in S^{m-1}$. munosabat o`rinli bo`lsa, ν -tartibli bistoxastik operator deyiladi.

ν -tartibli bistoxastik **B** operatorni $\mathbf{B}^{[\nu]}$ ko`rinishida belgilaymiz.

3.BISTOXASTIK OPERATORLARNING BA`ZI XOSSALARI

Teorema 2.1. *Har qanday* ν *-tartibli* $B^{[\nu]}$ *bistoxastik operator yakabianining simpleks markazi* $m^{-1} = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$*dagi qiymati* $J(m^{-1}) = \nu \cdot B$ *ko`rinishda bo`ladi. Bu yerda* B *bistoxastik matritsa.*

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Isbot: aytaylik S stoxastik operator bo`lsin. U holda $S$ operatorning yakabiani quyidagi ko`rinishda bo`ladi:

$$J(x)=\begin{pmatrix} \frac{\partial\varphi_1(x)}{\partial x_1} & \frac{\partial\varphi_1(x)}{\partial x_2} & \cdots & \frac{\partial\varphi_1(x)}{\partial x_m} \\ \frac{\partial\varphi_2(x)}{\partial x_1} & \frac{\partial\varphi_2(x)}{\partial x_2} & \cdots & \frac{\partial\varphi_m(x)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\varphi_m(x)}{\partial x_1} & \frac{\partial\varphi_m(x)}{\partial x_2} & \cdots & \frac{\partial\varphi_m(x)}{\partial x_m} \end{pmatrix}.$$

Biz $J(m^{-1})$ matritsaning har bir satr va har bir ustuni yig`indisi $\nu$ ga tengligini ko`rsatamiz.

***Satr bo`yicha***: Bizga ma`lumki $m^{-1}\in S_{>}^{m-1}$ ekanligidan, $\varphi_k(x)$ funksiyaning xususiy hosilalari uchun

$$(\varphi_k(x))'_{x_j}=\sum_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_j(i_1,i_2,\ldots,i_\nu)P_{i_1i_2\ldots i_\nu,k}\frac{x_1^{\tau_1}x_2^{\tau_2}\ldots x_m^{\tau_m}}{x_j}$$

tenglik o`rinli bo`ladi. Natijada yakabianning simpleks markazi $m^{-1}=\left(\frac{1}{m},\frac{1}{m},\ldots,\frac{1}{m}\right)$ dagi qiymati quyidagi ko`rinishda bo`ladi:

$$J(\mathrm{m}^{-1})=\frac{1}{m^{\nu-1}}\cdot\begin{pmatrix} \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_1P_{i_1i_2\ldots i_\nu,1} & \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_2P_{i_1i_2\ldots i_\nu,1} & \cdots & \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_mP_{i_1i_2\ldots i_\nu,1} \\ \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_1P_{i_1i_2\ldots i_\nu,2} & \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_2P_{i_1i_2\ldots i_\nu,2} & \cdots & \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_mP_{i_1i_2\ldots i_\nu,2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_1P_{i_1i_2\ldots i_\nu,\mathrm{m}} & \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_2P_{i_1i_2\ldots i_\nu,\mathrm{m}} & \cdots & \sum\limits_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_mP_{i_1i_2\ldots i_\nu,\mathrm{m}} \end{pmatrix}.$$

Hosil bo`lgan matritsaning ixtiyoriy satr yig`indisi ν ga teng ekanligini ko`rsatamiz. Ixtiyoriy $j\in N_{\le m}$ - satr yig`indisi uchun

$$J(\mathrm{m}^{-1})=\frac{1}{m^{\nu-1}}\cdot\left(\sum_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_1P_{i_1i_2\ldots i_\nu,\mathrm{j}}+\sum_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_2P_{i_1i_2\ldots i_\nu,\mathrm{j}}+\ldots+\sum_{i_1,i_2,\ldots,i_\nu=1}^{m}\tau_mP_{i_1i_2\ldots i_\nu,\mathrm{j}}\right)=$$

$$=\frac{1}{m^{\nu-1}}\sum_{i_1,i_2,\dots,i_\nu=1}^{m}P_{i_1i_2\dots i_\nu,j}\cdot(\tau_1+\tau_2+\dots+\tau_m).$$

Lemma 2 [6] ga ko`ra $\sum_{i_1,i_2,\dots,i_\nu=1}^{m}P_{i_1i_2\dots i_\nu,j}=m^{\nu-1}$ tenglik o`rinli ekanligidan

$$J(m^{-1})=\frac{1}{m^{\nu-1}}\cdot m^{\nu-1}\cdot(\tau_1+\tau_2+\dots+\tau_m)=\nu.$$

Demak $J(m^{-1})$ matritsaning har bir satr elementlari yig`indisi ν ga teng ekan.

Ustun bo`yicha:

Biz $J(S^{m-1})$ matritsaning har bir ustuni yig`indisi $\nu$ ga tengligini ko`rsatamiz. $J(x)$ matritsaning ixtiyoriy $j\in N_{\le m}$ - ustun yig`indisi uchun

$$\frac{\partial\varphi_1(x)}{\partial x_1}+\frac{\partial\varphi_2(x)}{\partial x_1}+\dots+\frac{\partial\varphi_m(x)}{\partial x_1}=\frac{\partial(\varphi_1(x)+\varphi_2(x)+\dots+\varphi_m(x))}{\partial x_1}=$$

$$=\frac{\partial(x_1+x_2+\dots+x_m)^{\nu}}{\partial x_1}=\nu(x_1+x_2+\dots+x_m)^{\nu-1}=\nu\left(\underbrace{\frac{1}{m}+\frac{1}{m}+\dots+\frac{1}{m}}_{m\ \ ta}\right)^{\nu-1}=\nu$$

tenglik o`rinli bo`ladi. $x_1+x_2+\dots+x_m=1$ ekanligidan, $J(m^{-1})$ matritsaning har bir ustun elementlari yig`indisi $\nu$ ga teng bo`lishi kelib chiqadi.

Biz ixtiyoriy ν-tartibli bistoxastik operator yakabiani $J(x)$ ning $m^{-1}=\left(\frac{1}{m},\frac{1}{m},\dots,\frac{1}{m}\right)$dagi qiymati har bir satr va har bir ustun elementlari yig`indisi $\nu$ ga teng bo`lgan matritsa ekanligini isbotladik, bu esa $J(m^{-1})=\nu\cdot B$ ko`rinishda ekanligini bildiradi. +

Natija 2.1. *Har qanday ν-tartibli $S^{[\nu]}$ stoxastik operator yakabianining simpleksdagi ko`rinishi $J(S^{m-1})=\nu\cdot B$ ko`rinishda bo`ladi. Bu yerda B stoxastik matritsa.*

Teorema 2.2. $B:S^{m-1}\to S^{m-1}$ *bistoxastik operator uchun* $m^{-1}=\left(\frac{1}{m},\frac{1}{m},\dots,\frac{1}{m}\right)$ *qo`zg`almas nuqta itaruvchi emas.*

Isbot. Teskari faraz qilaylik , m^{-1} qo`zg`almas nuqta *itaruvchi* qo`zg`almas nuqta bo`lsin. Bundan kelib chiqadiki m^{-1} nuqta uchun shunday

$$U_\alpha(m^{-1})=\left(x=(x_1,x_2,\dots,x_m)\in S^{m-1}\mid\rho(x,m^{-1})<\alpha\right)\qquad(3.1)$$

atrof mavjudki, bu atrofdan olingan ixtiyoriy $x^{(0)} \in U_{\alpha}\left(m^{-1}\right)$ uchun

$$x^{(1)} = B\left(x^{(0)}\right), x^{(2)} = B\left(x^{(1)}\right) = B^{(2)}\left(x^{(0)}\right), \ldots, x^{(k)} = B\left(x^{(k-1)}\right), \ldots \qquad (3.2)$$

ko`rinishda hosil qilingan ketma ketlikdan shunday $k \in N$ -had topiladiki,

$$x^{(k)} \in U_{\alpha}\left(m^{-1}\right) \text{ va } x^{(k+i)} \notin U_{\alpha}\left(m^{-1}\right), \quad i = 1, 2, 3, \ldots \qquad (3.3)$$

bo`ladi. Ushbu

$$U_{\alpha}\left(m^{-1}\right) = \left(x = \left(x_1, x_2, \ldots, x_m\right) \in S^{m-1} \mid \rho_{\max}\left(x, x^*\right) < \alpha \right)$$

atrofni qaraylik. Ko`rishimiz mumkinki

$$U_{\alpha}\left(m^{-1}\right) \subset U_{\alpha}(\mathrm{m}^{-1})$$

bo`ladi . Demak ixtiyoriy $x^{(0)} \in U_{\alpha}\left(m^{-1}\right)$ uchun ham ( 3.2) ko`rinishda hosil qilingan ketma ketlikdan shunday $k \in N$ -nomer topiladiki,

$$x^{(k)} \in U_{\alpha}\left(m^{-1}\right) \text{ va } x^{(k+i)} \notin U_{\alpha}\left(m^{-1}\right), \quad i = 1, 2, 3, \ldots \qquad (3.3)$$

bo`ladi. $x^{(k)} \in U_{\alpha}\left(m^{-1}\right)$ ekanligidan

$$\rho_{\max}\left(x^{(k)}, \mathrm{m}^{-1}\right) < \alpha \quad \Rightarrow \quad x_{[1]}^{(k)} - \frac{1}{m} < \alpha \quad \Rightarrow \quad x_{[1]}^{(k)} < \alpha + \frac{1}{m}$$

va $x^{(k+i)} \notin U_{\alpha}\left(m^{-1}\right), \quad i = 1, 2, 3, \ldots$ ekanligidan

$$\rho_{\max}\left(x^{(k+i)}, x^*\right) \geq \alpha \quad \Rightarrow \quad x_{[1]}^{(k+i)} - \frac{1}{m} \geq \alpha \quad \Rightarrow \quad x_{[1]}^{(k+i)} \geq \alpha + \frac{1}{m}, i = 1, 2, 3, \ldots$$

bo`lishi kelib chiqadi. Demak $x_{[1]}^{(k+i)} > x_{[1]}^{(k)}$ ekan, xususan $i = 1$ bo`lganda

$$x_{[1]}^{(k+1)} > x_{[1]}^{(k)} \qquad (3.4)$$

tengsizlik o`rinli ekan. Boshqa tomondan $B$ bistoxastik operator ekanligidan ya`ni bistoxastik operator ta`rifidan kelib chiqadiki

$$B^{k+1}\left(x^{(0)}\right) \prec B^k\left(x^{(0)}\right) \quad \text{ya`ni} \quad x_{[1]}^{(k+1)} \leq x_{[1]}^{(k)} \qquad (3.5)$$

(3.4) va (3.5) tengsizliklardagi ziddiyat farazimiz noto`g`ri ekanligini isbotlaydi. Demak $m^{-1} = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)$ qo`zg`almas nuqta *itaruvchi* emas. +

Quyidagi to`plamni qaraylik

$$M_\alpha = \left\{ x = \left(x_1, x_2, ..., x_m \right) \in S^{m-1} : x_{[1]} \le \alpha \right\}.$$

Teorema 2.3. B *bistoxastik operator uchun* $B\left(M_\alpha\right) \subseteq M_\alpha$ *munosabat o`rinli bo`ladi.*

Isbot: Aytaylik B bistoxastik operator va ixtiyoriy $x^{(0)} \in M_\alpha$ bo`lsin. Tabiiyki $x_{[1]}^{(0)} \le \alpha$ bo`ladi. Endi $B\left(x^{(0)}\right) = x^{(1)}$ deb belgilaylik. Bistoxastik operator ta`rifiga ko`ra $x_{[1]}^{(1)} \le x_{[1]}^{(0)}$ tengsizlik o`rinli. Bundan kelib chiqadiki $x_{[1]}^{(1)} \le \alpha$, ya`ni $B\left(x^{(0)}\right) \in M_\alpha$. +

Teorema 3.3. *Bistoxastik operator simpleksda* $m^{-1} = \left(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m} \right)$ *nuqtadan farqli tortuvchi qo`zg`almas nuqtaga ega emas.*

Isbot. Teskarsini faraz qilaylik, ya`ni $B$ bistoxastik operator biror $x^* \in S^{m-1}$ tortuvchi qo`zg`almas nuqtaga ega bo`lsin. U holda shunday

$$U_\varepsilon\left(x^*\right) = \left\{ x = \left(x_1, x_2, ..., x_m \right) \in S^{m-1} : \rho\left(x, x^*\right) < \varepsilon \right\}$$

atrof mavjudki, ixtiyoriy $x^{(0)} \in U\left(x^*\right)$ uchun (3.2) ko`rinishda hosil qilingan ketma-ketlikda $\lim_{n\to\infty} x^{(n)} = x^*$ bo`ladi. U holda shunday M_α to`plam mavjudki $U_\varepsilon\left(x^*\right) \cap M_\alpha \ne \varnothing$ va $x^* \notin M_\alpha$ bo`ladi. Bunda α ni $\alpha = x_{[1]}^* - \frac{\varepsilon}{2}$ ko`rinishda tanlashimiz mumkin. Endi ixtiyoriy $x^{(0)} \in U_\varepsilon\left(x^*\right) \cap M_\alpha$ uchun teorema 3.3 ga ko`ra $\lim_{n\to\infty} x^{(n)} \in M_\alpha$ bo`ladi, biroq $x^* \notin M_\alpha$ demak farazimiz noto`g`ri. bundan kelib chiqadiki, $C : S^{m-1} \to S^{m-1}$ bistoxastik operator tortuvchi qo`zg`almas nuqtaga ega emas.

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