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VAQT BOʻYICHA KASR TARTIBLI TOʻLQIN TENGLAMASI UCHUN BIR NOLOKAL MASALA HAQIDA

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Abstract: Ushbu maqolada vaqt oʻzgaruvchisi boʻyicha umumlashgan Xilfer kasr tartibli operatori ishtirok etgan toʻlqin tenglamasi uchun nolokal shartli chegaraviy masalaning bir qiymatli yechilishi tadqiq etilgan. Bunda oʻzgaruvchilarni ajratish usuli hamda kasr tartibli tenglama uchun Koshi masalasini yechish usullaridan foydalanilgan.

Keywords:. Xilfer kasr tartibli hosilasi; toʻlqin tenglamasi; Koshi masalasi; Furye usuli.

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ON A NONLOCAL PROBLEM FOR TIME-FRACTIONAL WAVE EQUATION

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Abstract: In this paper, a unique solvability of a problem with nonlocal condition for the wave equation involving generalized Hilfer fractional derivative has been proved. The method of a separation of variables and the solution of the Cauchy problem are key tools of investigation.

Keywords:. Hilfer fractional derivative; wave equation; Cauchy problem; Fourier method.

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ОБ ОДНОЙ НЕЛОКАЛЬНОЙ ЗАДАЧЕ ДЛЯ ВОЛНОВОГО УРАВНЕНИЯ ДРОБНОГО ПОРЯДКА ПО ВРЕМЕНИ

Алимов З.
Оллояров М.

Abstract: В этой статье доказана однозначная разрешимость задачи с нелокальными условиями для волнового уравнения с обобщенным дробным производным Хилфера. При этом использованы метод разделения переменных и решение задачи Коши..

Keywords:. Дробное производное Хилфера; волновое уравнение; задача Коши; метод Фурье..

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Ma'lumki, xususiy hosilali differensial tenglamalar fizika, mexanika, biologiya, kimyo va boshqa ko'plab amaliyotga bog'langan fanlardagi jarayonlarni matematik modellashtirishda muhim ahamiyat kasb etadi [1-4]. Oxirgi yillarda bunday matematik modellarni mukkamallashtirishda kasr tartibli hosilalar qatnashgan turli xususiy hosilali differensial tenglamalardan foydalanilyapti [5-6]. Bunga kasr tartibli analizning o'ziga xos usullari ishlab chiqilgani ham asos bo'lib xizmat qilmoqda [7-8]. Kasr tartibli hosilalar asosan vaqt o'zgaruvchisiga nisbatan qo'llanilayotganining ham o'ziga xos sabablari bo'lib, bu xotira effekti, g'ovak muhitlar bilan bog'liqdir [9]. Kasr tartibli hosilalarning ham turli tiplari kiritilgan bo'lib [10], ularning strukturasiga qarab turli maqsadlarda foydalanilyapti. Ushbu ishda qo'llanilgan umumlashgan Xilfer kasr tartibli hosilasi dastlab [11] ishda kiritilib o'rganilgan bo'lishiga qaramasdan, bu hosilani avvalroq kiritilgan, lekin ko'pchilikka ma`lum bo'lmagan Jrbashyan-Nersesyan operatorining xususiy holi sifatida olish mumkin [12].

Quyidagi

$$D_{0t}^{(\alpha,\beta)\mu}u(x,y,t) = U_{xx}(x,y,t) + U_{yy}(x,y,t) \qquad (1)$$

tenglamani $\Omega = \{(x,y,t) : 0 < t < T, 0 < x < 1, 0 < y < 1\}$ sohada tadqiq etamiz. Bu yerda α, β -shunday haqiyqiy sonlarki, $1 < \alpha, \beta \le 2$ va

$$D_{ot}^{(\alpha,\beta)}f(t) = I_{ot}^{\mu(2-\alpha)}\left(\frac{d}{dt}\right)^2 I_{ot}^{(1-\mu)(2-\beta)}f(t) \qquad (2)$$

α, β tartibli, $\mu \in [0,1]$ tipdagi umumlashgan Xilfer kasr tartibli operator [11], I_{ot}^{γ} esa

$$I_{ot}^{\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-z)^{\gamma-1}f(z)dz, \; t > 0 \qquad (3)$$

formula bilan aniqlanuvchi γ -tartibli Riman-Liuvill integral operatori [7].

(1) tenglama uchun Ω sohada quyidagi masalani taqdim etamiz:

Masala. (1) tenglamaning Ω sohadagi quyidagi shartlarni qanoatlantiruvchi regulyar yechimi topilsin:

$$u(0,y,t) = 0, \; u(1,y,t) = 0 \quad 0 \le y \le 1, 0 < t \le T, \qquad (4)$$

$$u(x,0,t) = 0, \; u(x,1,t) = 0 \quad 0 \le x \le 1, 0 < t \le T, \qquad (5)$$

$$\lim_{t \to 0} I_{0t}^{(1-\mu)(1-\beta)}u(x,y,t) = u(x,y,\xi) + \varphi(x,y), \; 0 \le x \le 1, \quad 0 \le y \le 1 \quad 0 < \xi \le T$$

(6)

$$\lim_{t \to 0} \frac{d}{dt} I_{ot}^{(1-\mu)(2-\beta)}u(x,y,t) = \psi(x,y), \; 0 < x < 1, \; 0 < y < 1, \qquad (7)$$

Bu yerda $\varphi(x,y), \psi(x,y)$ - berilgan funksiyalar.

Ta'rif. (1) tenglamaning Ω sohadagi **regulyar yechimi** deb shunday $u(x,y,t)$, funksiyaga aytiladiki u

$$D_{o_t}^{(\alpha,\beta)\mu}u(x,y,t)\in C(\Omega),\ u_{xx}(x,y,t),\ u_{yy}(x,y,t)\in C(\Omega),\ t^{(1-\mu)(2-\beta)}u(x,y,t)\in C(\overline{\Omega})$$

regulyarlik shartlarini hamda Ω sohada (1) tenglamani qanoatlantiradi.

Masalani o'zgaruvchilarni ajratish usuli bilan tadqiq etamiz. Bunda masala yechimi

$$u(x,y,t)=\sum_{m,n=0}^{\infty}U_{m,n}(t)\sin(n\pi x)\sin(m\pi y) \qquad (8)$$

ko'rinishda qidiriladi, bu yerda $U_{m,n}(t)$ noma'lum funksiyalar bo'lib, ularni topish uchun keyinchalik mos nolokal shartli masalaga kelinadi. Buning uchun (8) ni (1) tenglamaga va (6), (7) boshlang'ich shartlarga qo'yamiz va t o'zgaruvchiga nisbatan quyidagi nolokal masalani olamiz:

$$\begin{cases} D_{0t}^{(\alpha,\beta)\mu}U_{m,n}(t)+\lambda_{m,n}U_{m,n}(t)=0, \\ \lim\limits_{t\to 0} I_{0t}^{(1-\mu)(1-\beta)}U_{m,n}(t)=U_{m,n}(\xi)+\varphi_{m,n}, \\ \lim\limits_{t\to 0}\dfrac{d}{dt}I_{0t}^{(1-\mu)(1-\beta)}U_{m,n}(t)=\psi_{m,n}. \end{cases} \qquad (9)$$

Bu yerda $\lambda_{m,n}=(n\pi)^2+(m\pi)^2,\ m,n\in\mathbf{N},\ \varphi_{m,n}\ va\ \psi_{m,n}$ lar esa mos ravishda $\varphi(x,y)$ va $\psi(x,y)$ funksiyalarning Furye koeffitsiyentlaridir.

Kasr tartibli hosilalar ishtirok etgan to'lqin tenglamalari uchun masalalar [13-14] ishlarda ham tadqiq etilgan.

Agar $U_{m,n}(\xi)$ ni vaqtincha ma'lum deb tursak, (9) masalaning yechimi dastlab

$$U_{m,n}(t)=\left[U_{m,n}(\xi)+\varphi_{m,n}\right]t^{\gamma-1}E_{\delta,\gamma}\left(-\lambda_{m,n}t^{\delta}\right)+\psi_{m,n}t^{\gamma-2}E_{\delta,\gamma-1}\left(-\lambda_{m,n}t^{\delta}\right) \quad (10)$$

ko'rinishida yoziladi. Bu yerda

$$\gamma=\beta+\mu(2-\beta),\ \delta=\beta+\mu(\alpha-\beta),\ E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\ -$$

ikki karrali Mittag-Leffler funksiyasi [8].

(10) da $t=\xi$ desak

$$U_{m,n}(\xi)=\left[U_{m,n}(\xi)+\varphi_{m,n}\right]\xi^{\gamma-1}E_{\delta,\gamma}\left(-\lambda_{m,n}\xi^{\gamma}\right)+\psi_{m,n}\xi^{\gamma-2}E_{\delta,\gamma}\left(-\lambda_{m,n}\xi^{\gamma}\right)$$

hosil bo'ladi. Bu yerda esa $U_{m,n}(\xi)$ ni quyidagicha topamiz:

$$U_{m,n}(x)=\frac{1}{1-x^{g-1}E_{d,g}\left(-m_{m,n}x^{d}\right)}\left[j_{m,n}x^{g-2}E_{d,g}\left(-m_{m,n}x^{d}\right)+y_{m,n}x^{g-2}E_{d,g-1}\left(-m_{m,n}x^{d}\right)\right]$$

(11)

Ta'kidlash kerakki, $x\neq 0$ bo'lgani uchun $1-x^{g-1}E_{d,g}\left(-m_{m,n}x^{g}\right)\neq 0$ barcha $d,g,m_{m,n}$ lar uchun bajariladi. Demak, (11) ni hisobga olib (10) ni quyidagicha yozib olish mumkin:

$$U_{m,n}(t)=j_{m,n}t^{g-1}E_{d,g}\left(-m_{m,n}t^{d}\right)+y_{m,n}t^{g-2}E_{d,g-1}\left(-m_{m,n}t^{d}\right)+$$
$$+\frac{j_{m,n}x^{g-1}E_{d,g}\left(-m_{m,n}x^{d}\right)+y_{m,n}x^{g-2}E_{d,g-1}\left(-m_{m,n}x^{d}\right)}{1-x^{g-1}E_{d,g}\left(-m_{m,n}x^{d}\right)}. \qquad (12)$$

(8) formula bilan berilgan cheksiz qatorning teksi yaqinlashishini isbotlash uchun$U_{m,n}(t)$ baholab olamiz. Buning uchun

$$\left|E_{a,b}(z)\right|\leq\frac{c}{1+|z|} \qquad (13)$$

bahodan foydalanamiz [8].

$$\left|U_{m,n}(t)\right|\leq\left|j_{m,n}\right|\cdot\frac{\left|t^{g-1}\right|\cdot c}{1+\left|m_{m,n}t^{d}\right|}+\left|y_{m,n}\right|\cdot\frac{c\left|t^{g-2}\right|}{1+\left|m_{m,n}t^{d}\right|}+\left|j_{m,n}\right|\cdot\frac{\left|x^{g-1}\right|\cdot c}{1+\left|m_{m,n}x^{d}\right|}+\left|y_{m,n}\right|\cdot\frac{c\left|x^{g-2}\right|}{1+\left|m_{m,n}x^{d}\right|}$$

yoki

$$\left|U_{m,n}(t)\right|\leq\frac{c_1}{\left|m_{m,n}\right|}\cdot\frac{\left|j_{m,n}\right|}{\left|t^{d+1-g}\right|}+\frac{c_2}{\left|m_{m,n}\right|}\cdot\frac{\left|y_{m,n}\right|}{\left|t^{d+2-g}\right|}. \qquad (14)$$

(14) ni hisobga olsak (8) dan

$$\left|u(x,y,t)\right|\leq\sum_{m,n=0}^{\infty}\frac{\left|j_{m,n}\right|}{\left|m_{m,n}\right|}\frac{c_1}{\left|t^{d+1-g}\right|}+\frac{\left|y_{m,n}\right|}{\left|m_{m,n}\right|}\frac{c_2}{\left|t^{d+2-g}\right|} \qquad (15)$$

ni olish mumkin. $u_{xx}(x,y,t)$, $u_{yy}(x,y,t)$funksiyalarga mos keluvchi cheksiz qatorlar uchun

$$\left|u_{xx}(x,y,t)\right|\leq\sum_{m,n=0}^{\infty}\left(\frac{c_3}{\left|t^{d+1-g}\right|}\left|j_{m,n}\right|+\frac{c_4}{\left|t^{d+2-g}\right|}\left|y_{m,n}\right|\right),$$

$$\left|u_{yy}(x,y,t)\right|\leq\sum_{m,n=0}^{\infty}\left(\frac{c_3}{\left|t^{d+1-g}\right|}\left|j_{m,n}\right|+\frac{c_4}{\left|t^{d+2-g}\right|}\left|y_{m,n}\right|\right) \qquad (16)$$

baho kerak bo'ladi. (15) va (16) da Parseval tenglidan foydalansak [1]

$$\left|u(x,y,t)\right|\leq c_5+\|j\|_2^2+\|y\|_2^2,\quad \left|u_{xx}(x,y,t)\right|\leq c_6+\|j'_x\|_2^2+\|y'_x\|_2^2,$$

$$\left|u_{yy}\left(x,y,t\right)\right| \text{ J } c_7+\left\|j_y^{'}\right\|_2^2+\left\|y_y^{'}\right\|_2^2 \qquad (17)$$

ni olamiz. Bu esa $u\left(x,y,t\right)$ *va* $u_{xx}\left(x,y,t\right)$ funksiyalarga mos keluvchi cheksiz qatorlarning tekis yaqinlashishini isbotlaydi.

Masala yechimining yagonaligi fazoviy o‘zgaruvchilar bo‘yicha olingan sistemalarning to‘la ortonormal basis tashkil etishidan kelib chiqadi. Demak, quyidagi tasdiq o‘rinli:

Teorema. *Agar*

$$\varphi(x,y),\,\psi(x,y)\in C\left([0,1]\times[0,1]\right),$$

$$\varphi_x(x,y),\,\psi_x(x,y)\in L_2\left((0,1)\times(0,1)\right),\,\varphi_y(x,y),\,\psi_y(x,y)\in L_2\left((0,1)\times(0,1)\right)$$

bo‘lsa, u holda masala yagona yechimga ega va u (8) ko`rinishda topiladi.

FOYDALANILGAN ADABIYOTLAR RO‘YXATI:

1. Courant, R., Hilbert, D. (1962). Methods of Mathematical Physics, vol. II, New York: Wiley-Interscience.
2. Gershenfeld, N. (1999). The Nature of Mathematical Modeling (1st ed.), New York: Cambridge University Press.
3. Wazwaz, Abdul-Majid (2009). Partial Differential Equations and Solitary Waves Theory. Higher Education Press.
4. Roubíček, T. (2013). Nonlinear Partial Differential Equations with Applications. International Series of Numerical Mathematics, vol. 153 (2nd ed.), Basel, Boston, Berlin: Birkhäuser.
5. Uchaikin V.V. (2013). Fractional Derivatives for Physicists and Engineers. Vol.1. Background and Theory. Springer, Berlin, Heidelberg.
6. Uchaikin V.V. (2013). Fractional Derivatives for Physicists and Engineers. Vol.2. Applications. Springer, Berlin, Heidelberg.
7. Kilbas A.A., Srivastava H.M., Trujillo J.J. (2006). Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.
8. Podlubny I. (1999). Fractional Differential Equations. Academic Press, San Diego.
9. Nigmatullin R.R. (1984). On the theory of relaxation with “remnant” memory. Phys. Status Solidi B, 124(1), pp.389-393.
10. De Oliveira, E.C.; Tenreiro Machado, J.A. (2014). A review of definitions for fractional derivatives and integral. Math. Probl. Eng., 238459.
11. Bulavatsky V.M. (2014). Closed form of the solutions of some boundary problems for anomalous diffusion equation with Hilfer’s generalized derivative. Cybernetics and Systems Analysis. 30(4), pp. 570-577.

12. Dzhrbashyan M. M., Nersesyan A. B. (1968). Fractional Derivatives and the Cauchy Problem for Fractional Differential Equations, Izv. Akad. Nauk Armyan. SSR. 3, No 1 (1968), 3-29.

13. Alimov Z. S. (2022). On a boundary-value problem for the degenerate wave equation in a prismatic domain. Bull. Inst. Math., Vol. 5, No3, pp. 36-40.

14. David J., Nolte A., Sherman J. (2018). A Boundary-Value Problem for 3-D Fractional Wave Equation with Singularity. Bull. Inst. Math., Vol. 1, No2, pp. 28-52.